# An Introduction to Curriculum Zero

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# Contents

# Front Matter

*Mrs Shakti Gattegno, whose husband Caleb (1911-1988) called his approach "the subordination of teaching to learning", said "Dr Gattegno's approach and the realm of electronic technology need to be blended in a manner that neither one would end up eclipsed by the other. Dr Benson is sensitive to this requirement."* Maths targets can be achieved by teaching algebra first. Press Release, Sociality, 2005

*For over forty years, the Algebra Project has been working at the forefront of a civil rights struggle against a root cause of racial inequity in the United States: math education. Bob Moses (1935-2021), its founder, reviewing the Tizard approach in 2007, said "In our country we have an education movement which is University based and school affiliated that holds itself accountable for three types of University products: tests and exams, instructional materials, and instruction for future teachers. However, successful intervention also requires holding ourselves accountable for the students who graduate from our high schools. That is why there is a need for organization to bring together parents, young people, mentors, mathematicians and teachers. It is said that it takes a village to raise a child, true enough, but in the twenty-first century's global technology it takes a global village to make and deliver on the promise of a quality education for every child."* Ian Benson, The Primary Mathematics: Lessons from the Gattegno School, Lambert Academic, page 8, 2011

*"Some schools have developed schemes and use programmes that first stress the concrete, abstract and algebraic aspects of mathematics, and then apply them to understanding number and calculation. For example, 'Cuisenaire' resources were used very effectively in one school visited by the panel, where the defining criteria for success were undoubtedly the enthusiasm and expertise of the head teacher and the staff for this approach."* Sir Peter Williams, Independent Review of Mathematics Teaching in the Early Years and Primary Schools, page 61, 2009

*"The creativity of the teacher is related to, interconnected with, the creativity of the child. The more the child is becoming creative, the more the teacher becomes creative. And the more the teacher is creative, the more the children are creative. It is interconnected. But, when we have people who help teachers, who are consultants, coach - as we used to set out - the coach also should*

*be creative. But, the creativity of the coach should be interconnected with the creativity of the teacher. We work with what the teacher does. So we don't have classes who are looking so much one like another. We have differences because when there is creativity of the teacher, there is creativity of the class as a whole."* Madelaine Goutard, interview with Ian Benson, 2009

*" The National Association of Educational Inspectors Advisers and Consultants grew in membership and professional activity in the opening years of the 21st Century and contracted Sociality to enhance our website and seize the growing opportunities provided by the internet to improve our communications and profile. Their technical expertise and collaborative approach proved invaluable and considerable progress was achieved with their support. In addition, Sociality's wider philosophy dovetailed well with our strong commitment to education and children's services."* John Chowcat, General Secretary NAEIAC 2000 to 2013

*"I have very much enjoyed Conceptual Mathematics. I have lent it to two students this year, one of who enjoyed it so much he bought his own copy (the same student started a Haskell club: he's very keen on this sort of thing). I wish there was a way to introduce the perspective CM offers into the general curriculum. I'm interested in any thoughts/developments you have to share here".* Teacher, English Mathematics School, 2015

*"After an initial successful 8 week trial in Spring 2015, I adopted the algebraFirst™ approach to teaching number to my class of Year 1 children in September 2016. The project is part of the work led by Professor Ian Benson, facilitator for Sociality Mathematics, a CPD Network. His robust trials have already seen many successes and gains in children's mathematical understanding of concepts such as fractions, algebra and the use of the four operations. The algebraFirst™ approach involves using Cuisenaire rods to teach these concepts during daily mathematics lessons using Gattegno's textbook 1: 'Mathematics with Numbers in Colour'.* Rachel Rudge, Mind the Gap Both Ways: How Teaching Informs Research Decisions, in James Underwood (editor), Kaleidoscope Education Research Conference, University of Cambridge, 2016

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# Chapter 1

# Introduction

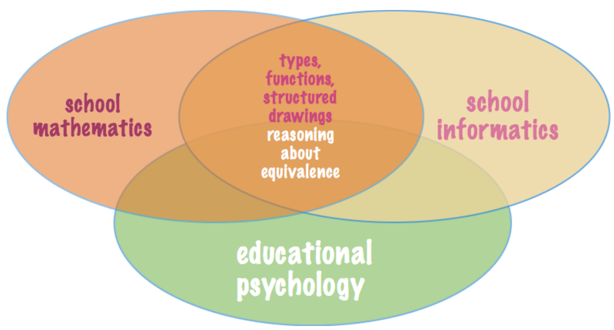


Figure 1.1: The landscape for improvement in school mathematics

This website, e-pub and pdf book (downloadable below) brings together influential research on the reform of the content and way of teaching mathematics in the UK conducted by mathematicians, teachers and learners working in partnership in a network of schools.

The movement for an intellectually ambitious mathematics reform called for by Bob Moses in the Front Matter also has deep roots in the UK. It was here that Caleb Gattegno and his colleagues founded the Association of Teachers of Mathematics in 1952. They investigated a simple reconfiguration of the sequence of concepts found in the traditional curriculum and mandated by followers of the influential developmental psychologist Jean Piaget. They found that this reconfiguration would overcome many of the difficulties teachers and pupils face. Gattegno claimed that he could cover the traditional primary arithmetic curriculum in 18 months. Instead of encountering the four operations and fractions as functions, then algebra, in succeeding years, Gattegno introduced all these ideas together, initially for small numbers, at Key Stage 1 (Gattegno, 1986).

We now know from informatics and mathematics research that modelling the concept of number in this "object-oriented" way helps learners to think clearly and efficiently about number systems (Benson, 2011) (Cheng, 2022). And we know, from recent studies of how we learn, that Gattegno's pedagogy prefigured our modern understanding of how the brain itself works (Gattegno, 1987) (Young and Messum, 2011) (Dehaene, 2020).

The obstacles that centrally mandated curricula and disciplinary boundaries place before teachers and pupils are one reason for the persistent shortfall in performance by pupils on secondary transfer. We call the new landscape for mathematics education, illustrated in Figure 1.1, Curriculum Zero. It is time for government, teachers, educationists and parents to work together to realise it (Financial Times, 2022).

## 1.1 About us

Sociality Mathematics CIC partners on mathematics school improvement with teachers, researchers and schools in California, New York, England and Wales. We are a social enterprise, asset-locked to Churchill College in the University of Cambridge. This website describes some of our influential partnerships and our approach to co-design of school curricula with teachers and learners. We are grateful to the authors for permission to reproduce the material in this digest.

To date we have worked with over 30 stakeholder schools. This target was set with Dick Tizard, a founding Fellow at Churchill and pioneer of university outreach. Described by Lord Broers, a former vice-chancellor of Cambridge University, as Cambridge's most significant Senior Tutor of the post-war years, Dick Tizard promoted "access" long before the term was coined.

The picture shows Dick with sometime members of the College Council discussing the progress of the project in 2005.

## 1.2 The initial brief

Our initial assignment was to re-evaluate Gattegno's proposal for mathematics curriculum reform for Charles Clarke, then Secretary of State for Education. The SoS wanted to get better value from the government's substantial investment in computer technology in schools. We were instructed by the DfES Innovation Unit not to create a new method of teaching, rather we were to re-evaluate a long standing initiative, assess the barriers to diffusion, and propose how new technologies for professional development of teachers might overcome these obstacles.

That work was carried out for the North West Leicestershire Primary Strategy Learning Network parents.sociality.tv.

These early years were written up in articles in Prospect Magazine and in the

Figure 1.2: Julian Filochowski, Dick Tizard, Michael Smyth and Ian Benson

alumni magazine of the University of Cambridge Computer Laboratory: Letter from Whitehall and Can Computer Science Rescue Mathematics Reform?

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# Chapter 2

# Classroom and desk based research

The initial work, advertised on the DfES website came to the attention of teachers and senior leaders at Stockland School, Honiton, Devon. With help from the North West Leicestershire Tizard network they adapted Gattegno's curriculum on a whole school basis from 2005-2020. In 2020 the school was absorbed into a multi-academy trust which chose to standardise on traditional teaching.

Figure 2.1 shows the extent of our classroom based research: from the initial reach of 24 students for 10 hours in N. W. Leics to the wholesale adaption of the Cuisenaire-Gattegno approach across Key Stage 1. After 2014 schools were able to take advantage of changes in the 2014 English national curriculum. These changes gave a statutory entitlement to the study of all four operations and fractions as functions for small numbers at Key Stage 1. Unfortunately, influential advice subsequently given to schools by the Department for Education, and endorsed in Ofsted inspection training, has silently rowed back from these obligations.

In Section 2.1 we reproduce a report by Caroline Ainsworth in Devon. An earlier version was published on the website of the National Centre for Excellence in the Teaching of Mathematics (NCETM). We have added captions to her videos. In Section 2.3 we collate articles written by teachers in Bexleyheath. These were published in the Mathematical Association Primary Mathematics journal. The Bexleyheath teachers took on the development of Caroline's initiative with colleagues in Leeds, Ipswich and Lambeth. In Section 2.5 we report on our comprehensive statistical analysis of the Cuisenaire-Gattegno research literature. In Section 2.6 we report on a 2 year longitudinal educational psychology study that contrasts a control school with an experimental school that adapted Gattegno's teaching materials in an 80 unit intervention over two years at Key Stage 1.

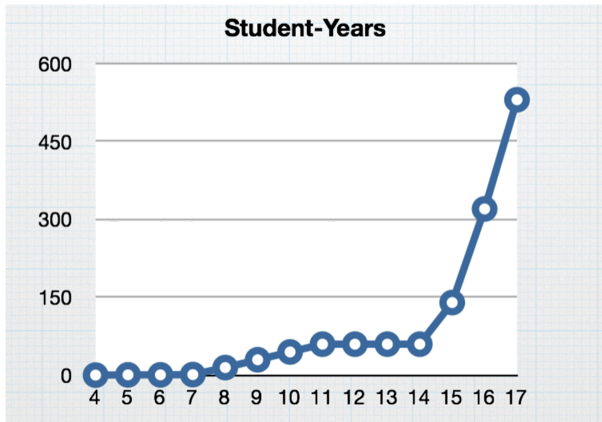



Figure 2.1: Classroom reach in student-years from 2004 to 2017

## 2.1 Case studies of teachers' professional development journeys

In 2005 Caroline Ainsworth, an early teacher in the Tizard network (2005-7), became interested in the use of Cuisenaire rods as a tool for teaching and learning mathematics. This led her to the work of Caleb Gattegno and Madeleine Goutard which she began to read extensively and which informed a series of investigations into her own teaching, her children's learning and the nature of mathematics.

As such hers is a rich and complex example of professional development which explores the interrelation between theory and practice.

In a unique and innovative collection of materials published on the NCETM website and reproduced in Section 2.2 she set out:

- an article about her work and ideas. She comments on a collection of videos she filmed of children in her school working on mathematics.
- a filmed discussion with Pete Griffin (NCETM SW regional coordinator)
- samples of her children's mathematical writing

This exercise was not intended to serve as an instruction manual. It is simply a story of one teacher's professional development which NCETM hoped would inspire and stimulate teachers to engage in their own research and professional development.

Although a suggested order has been offered these materials are flexible and can be used in a variety of ways and in (almost) any order. In the end the usefulness of these materials will rest in the extent to which they prompt teacher's own investigations. (While watching some of the video clips and working with some of the related written material you might find it useful to have a set of Cuisenaire rods to hand).

## 2.2 Foreword by Caroline Ainsworth

My aim in producing the materials in this section has been to try to capture my research process in all its complexity, rather than present polished findings. In order to do this effectively I felt that I needed to focus on something specific that I was trying to find out more about - something which seems to be important, but I don't yet fully understand.

I have settled on the teaching of fractions, with which I have already had some success, enough to become aware of just how much more the children are capable of. Therefore, the materials which you will find here (including video clips of children of various ages in my school working on mathematics) act as a diary indicating a cycle of:reading the theory and reflecting: translating my understanding into teaching; studying the children's responses; returning to the theory and so on.

Having been working in this way now for over year, this will be a snapshot of the process rather than showing the whole of it from the beginning. This will make it harder for you, the viewer, but that might be a good thing!

Rather than aiming to show a finished product or conclusion, it is this research cycle which I am trying to capture.

My research is in pursuit of understanding and replicating the fluent, complex expressions Goutard's children wrote at such a young age which included astonishing mastery of fractions (see Figure 2.2).

Already, the work my children are producing in their 'free writing' shows similarities with Goutard's children's work, (mine at a later age – mainly due to my lack of ambition with my teaching!) but I need now to analyse their writing more systematically, looking for specific features, in order to understand it better.

### 2.2.1 Filmed discussion

Here Caroline writes about her work and outlines some of the key ideas from Madeleine Goutard's writing which have inspired her. This article also includes video extracts from Caroline's classroom and samples of her children's written work to illustrate her approach.

***Improving mathematics teaching through studying the writing of Caleb Gattegno and Madeleine Goutard***

***Background***

Four years ago (in 2006) I worked through Caleb Gattegno's 'Numbers in Colour' (1957) text books in the hope of improving my own understanding, and therefore

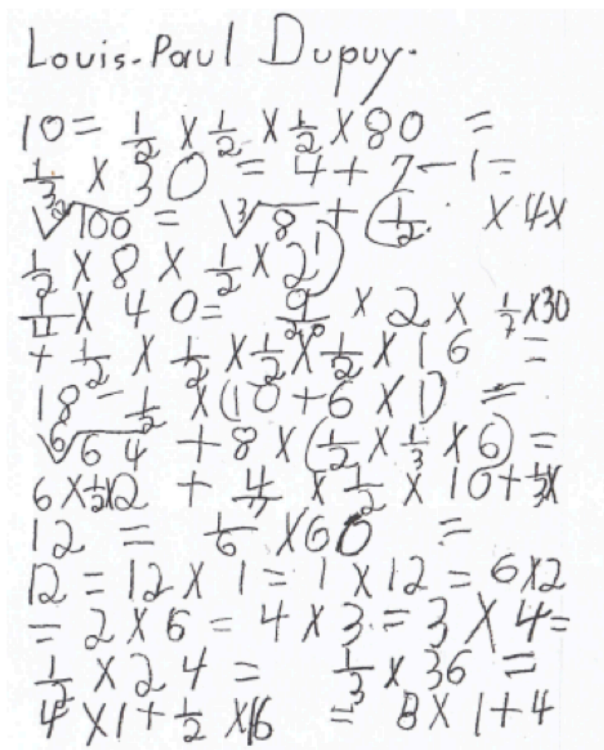



Figure 2.2: Grade 1 Student aged 6, after four months at school. page 177 [@xie2018]

my teaching, of mathematics. Written as a dialogue with the learner, Gattegno revealed to me a fascinating subject, very different from the one I was taught at school! Most importantly, everything I learnt I felt I had discovered for myself.

Hoping to give my children the same excitement of discovery and power over the subject, I initially used the text books with a class of year 3 and 4 children. From my very first attempts to teach with Gattegno's approach, I knew that the children were thinking differently about the subject: they had a new sense of purpose, discovery and power, and were gaining greater flexibility and creativity with calculation. I then came across the work of Madeleine Goutard (in turn influenced by Gattegno) who had interpreted this approach with classes of young children, with astonishing results.

*Mathematics and Children*, by Madeleine Goutard documents her work advising teachers in Quebec in the 1960s, and teaching mathematics to primary school children. Goutard describes and gives evidence of how the children she taught routinely gained mastery of addition, subtraction, division, multiplication and powers, by the age of 7 or before.

Inspired by seeing examples of her children's independent mathematical writing, I hoped to understand and perhaps even replicate the fluent, complex and creative mastery of expressions achieved by Goutard's children. I also wanted to understand the connection between this independent writing and the positive effects of Gattegno's approach which my children were already experiencing. However, it soon became clear to me that to understand her theory fully and to have any hope of reaching Goutard's level of achievement as a teacher, I would have to work in such a way that involved a constant reappraisal of my teaching; an opportunity to teach a wider age-range of children than just years 3 and 4, and a careful study of the texts.

I therefore began to teach mathematics in this way to all the children (Reception to Year 6) in my school. Now, three years into my research, encouraged by rising SATs results and children's confidence, I am convinced that this early mastery is not only possible but vital to children's later success in, and enjoyment of, mathematics. Whilst I do not claim to have reached Goutard's level of skill as a teacher, I have seen enough to share her belief (and that of Gattegno) that young children are capable of far more than is currently expected of them.

As a result of this research, I now understand that the key to this theory is to teach children, from their very first encounter with the subject, what Gattegno called the 'algebra' of operations, made possible through specific use of Cuisenaire rods. "the rods act as an algebraic model – for the algebra of arithmetic – making it possible to start with algebra instead of counting. It made sense to the students. They paid attention to the perceptible attributes and had very little to memorize. Therefore they could re-invent, easily and on every occasion, what was needed to solve a problem, and did not worry they might forget facts only held in their memory." p. 3 (Gattegno, 1987).

The Cuisenaire rods are used as a way of presenting what can be generalized about an number and number operations. In this way children become aware of addition as commutative and associative, division as repeated subtraction, multiplication as repeated addition, inverse operations, etc. before applying these 'rules' to any specific numbers. Importantly, the labeling of an operation with a mathematical sign can be agreed between teacher and children in the context of the shared experience of constructing an arrangement of rods. No further language is require as the arrangement itself and the act of constructing it contain all that is necessary to see what is distinct about an operation and what is shared with other operations. A shift of focus is all that is needed to re-label the same arrangement with a different mathematical sign. By encouraging independent 'free' mathematical writing, children then explore the algebra of operations (often as a series of transformations), working on all operations at once, and express their discoveries for themselves. In this way, the Cuisenaire rods can be used in all areas of number work, allowing children to constantly revisit their discoveries and build on previous learning.

What follows is an account of how my classroom-based research and my study of Goutard and Gattegno's theory of teaching have led me to these conclusions.

**Goutard's theory of teaching mathematics to young children**

"It is generally agreed that concrete experience must be the foundation of mathematics learning. When children find it difficult to understand arithmetic it is at once suggested that this is because it is too abstract; for small children the study is then simply reduced to the counting of objects. It seems to me that there has perhaps been too great a tendency to make things concrete and that perhaps the difficulties children experience spring from the fact that they are kept too much at the concrete level and are forced to use too empirical a mode of thought…

"The advantages of a material such as the one proposed by George Cuisenaire is that, paradoxically as it may seem, it enables children to reach an understanding of mathematical structures and frees them from the necessity to have recourse to a concrete support." p.2 (Goutard, 2017).

'Concrete' is defined as "existing in material form…, denoting thing as opposed to quality, state or action, not abstract" (Concise Oxford Dictionary p.195, 1982). Although the rods can be seen as concrete as they exist in material form, it seems to me there is a danger in using only their concrete property, particularly with very young children, thereby underestimating children's potential for abstract thought.

Goutard proposed that young children could move on quickly from the empirical mode of thought to making 'rational discoveries at the level of structures' (systematisation) and from there towards mastery of these structures.

I have found the following descriptions helpful in identifying the stages in my children's work.

**Empirical** – "based or acting on observation or experiment, not on theory;

regarding sense-data as valid information; deriving knowledge from experience alone.” Concise Oxford Dictionary 1982 p.315 “..one starts gleaning facts. This is done by trial and error, the results being accepted or rejected according to the criterion imposed on oneself. These facts are gathered at random, everybody gleaning what he can… Nevertheless they will only have been able to gather material. …The children have acquired more a technique than knowledge founded on reasons” p.6. (Goutard, 2017).

For example with respect to addition, children might be finding different trains of rods which are equivalent to another rod.

**Systematisation** – “.. to organise experience, to clarify facts so as to fill gaps if some are found, to propose groupings of some significance, in a word to invent sure means with which a thorough study of the situation could be undertaken.” p.8 (Goutard, 2017),

With addition, children may now be attempting to find out if they have found all the ways of partitioning a rod into smaller rods by grouping them in some way. They may use their knowledge that addition is commutative, or substitute two or more rods for an equivalent length to find new combinations of rods. They may begin to order their partitions according to some rule they have agreed.

**Mastery** – Goutard describes activities leading to mastery…. ”It is therefore towards a deeper understanding of the structures involved in these situations that the above discoveries take us. Every element or group of elements is seen to potentially contain the infinite set of which it is part, as soon as the dynamic link between the elements has(?) been noticed” p.18 (Goutard, 2017).

For example, in addition, children may discover that they can move from any one pair of complements to any other by adding the same amount to one rod as they have subtracted from the other.

These three phases of *empiricism*, *systematisation* and *mastery* are therefore crucial to Goutard’s theory of teaching mathematics and provided the focus for this part of my research. I needed to identify what I believed to be the characteristics of children working at each phase, understand its value, and find ways of moving children through the phases towards mastery.

**The Research Process**

My research follows a specific cycle:

From reading the texts:

- I identify something I want to understand better or questions I want to answer (in this case understanding Goutard’s 3 phases of children’s thinking);
- I translate this into a classroom activity and video groups of children working on it;
- I watch the clips back and look more closely at children’s responses both planned and unexpected;

- I return to the text with more practical understanding, and so begin the process again.

The aim of these videos and the research project as a whole is to try to show a continuous cycle of classroom research rather than present polished findings or recommend a specific pedagogy. Therefore, in what follows I have included a commentary to accompany each series of video clips showing how and why activities changed.

The videos that follow in Part 1 explore the value of the empirical phase of children's work. Discover how I tried to avoid them working at 'too empirical a mode of thought'

***Part 1 discussion*** *What is the value of the empirical phase of children's work and how can I avoid them working at 'too empirical a mode of thought'?*

In this sequence of clips, I use an activity in which I believe children are working empirically. The idea comes from Goutard's description of how she introduces fractions.

"In introducing fractions, the key to success is variety, change, diversity of points of view, and teaching as little as possible. Ordinarily I give only one fraction to the children, enough to furnish them with a terminology and written style; they discover the rest…We should engage in group discussion where possible answers are examined and the children should conduct experiments to determine if they are acceptable or if they must be modified. Only in this way can we truly learn. When we deny children the right to make mistakes, we do their work for them and tell them what they should be discovering for themselves" p.75 (Goutard, 1974)

I want to find out when children are given as little as Goutard suggests, how much will they be able to discover for themselves and whether the act of constructing the rest gives them the creativity and understanding that Goutard's children showed? I want to discover for myself what the value of the empirical phase is and how will it lead to systematisation?

***Class 2 (Year 2 and 3) Video sequence. Term 1 Naming fractions: "How many fit it?"***

In each of these clips (Cl2A to D), children are naming rods, taking turns to suggest and justify a name by comparing it with a rod already named. This activity allows me to hear the child's reasoning and to find out which comparisons they are using.

Clip C12A demonstrates the importance of comparison. When the first child says that the rod is 3, it could just be the third rod set out, when prompted he does say that it is three because three of the light green (named as 1) fit in. It is this same relationship which the second child now needs to label one third(?). After giving her time to think, the others prompt her by asking how many of the whites fit it and placing whites near the rod. This is enough to lead her to say “one third”. I recognize that it will be key when introducing fractions to the younger children for them to have an awareness that 3 means three of whatever is one. This awareness is allowing them to create names by comparison. The red is named as two thirds “because two of the third fit into it”. The green is named as three thirds. This is a start, but there seems to be a danger of them naming rods in succession using their length as measured by the white – pink as four thirds, yellow as five thirds, etc. But this will involve only limited comparison, so I try to “give the effort another twist and make another demand on the children” by setting out an order of rods to be named for this next activity p.7 (Goutard, 2017).

Clip Cl1B demonstrates another feature of the empirical phase. Children are free to be creative in their labeling with the result of more complexity emerging. On the white board the blue has been named as 1, so by having to name light green, they call it one third of 1, then dark green as two thirds of 1. White is named as one ninth, but red instead of being named as two ninths has been named as one third of two thirds of 1. Light green is then renamed as half of two thirds of one. By generating fraction names through making comparisons, the possibility of addition, multiplication and subtraction of fractions presents itself early and is accepted by the children as producing logical names. I notice that children will write an operation on a fraction – e.g. 2 x 1/3 before they will name it as 2/3. They are using something they are familiar with, they do not ‘invent’ the condensed name 2/3 – why should they when they have a perfectly useful and familiar alternative?

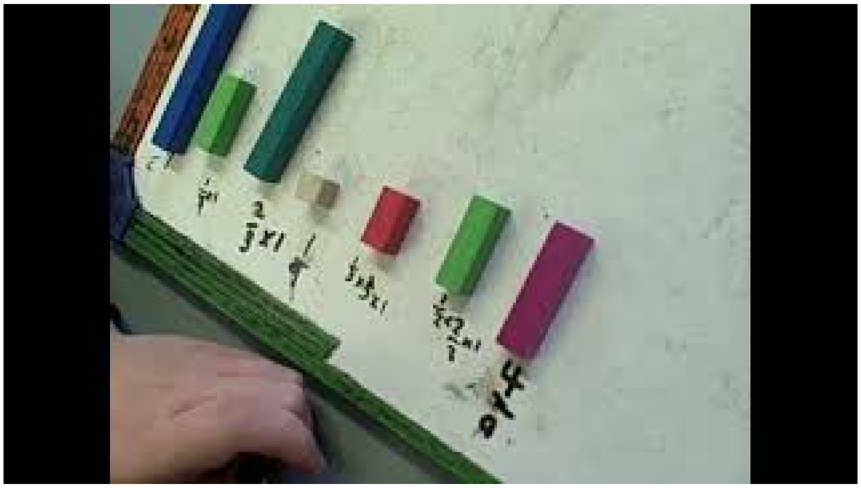


Figure 2.3: CI2B

Figure 2.4: CI2C

Clip Cl1C. Children seem to like complexity which leads naturally to equivalent fraction names. I notice they often add a new twist to the game themselves to satisfy their sense of fun at devising increasingly involved names. The first child starts to name the tan rod as eight fifths, but a 'more interesting name" is suggested. He then responds with 'four fifths plus four fifths', this uses the labeling of the act of placing rods end to end as 'plus' to naturally lead to addition of fractions. He now needs to be encouraged to leave the name eight fifths. The children are comfortable with the idea of a fraction having two or more names. An opportunity has occurred for moving towards systematisation. These fraction names could be saved and studied later when more similar examples have been gathered.

Figure 2.5: CI2D

Clip Cl1D. This activity takes place several weeks after the previous videos. I am wondering how I will know when the empirical phase is no longer valuable. Goutard warns "Nevertheless it would not be wise to shorten this phase of the investigation and throw oneself straight into a systematisation of facts, as the danger of drying up the minds of the pupils exists. This phase is the fountainhead of wealth, facility and technical knowledge needed for the future." p.7 (Goutard, 2017). So in this clip I am looking to see if there is anything 'fresh' in their comparisons; if this empirical work is still worthwhile. I notice that they are anticipating new names more quickly now, based on previous names (one child says says 'you've given away another name' as if it is obvious) but also that one child's reasons for naming refers to complements to one whole – she justifies the name 6/10 by referring to adding 4/10 or 2/5 to make a whole, then 7/10 as needing 3/10 more. This seems to justify fostering this creative approach as the children meet the need for handling fractions in different ways. Perhaps activities at the empirical phase which generate certain types of writings could be used in a more focused way to intentionally lead to systematisation.

"Since *mathematics only exists in the mind* and by the mind, the use of the material cannot be everything. ……True, notation rests upon the experience acquired through the manipulation of the rods but it can only come on its own *when the mind leaves the material and masters the significance of the manipulations*. Between the perceptive exploration of the material and notation, the important phase of the *conscious elaboration of the experience* takes place." p.35 (Goutard, 2017)

So, I need to give the children the opportunity to become freed from memory or the material itself. At this point, the use of the rods may be 'keeping the children at too empirical a mode of thought'. Goutard suggests that children should be given the chance to write their own mathematics daily, and is adamant that this should take place when the rods have been put away, hence allowing conscious elaboration of the experience to take place.

(see Samples of the children's writing Section 2.2.3)

This is useful for assessing individual progress, but by its very nature, this leads to children working at very different levels of thought at one time, so the challenge remains to plan activities allowing for this variety but also moving generally towards systematisation and mastery of structures. In the following activities devised for years 3 and 4, I am aiming to do this. I want to continue this freedom of expression and creativity but I also want the children to work at the level of structures, by seeing patterns and numerical relationships, which can still be verified by comparison to other lengths already named. I do not want them to use a 'rule' without recourse to their own logic. This may point to a danger of imposing a 'mastery of structure' upon children who have not discovered it for themselves therefore cannot verify it against their own understanding.

***Class 3 (Years 4 and 5) Video sequence "If that's…. then this would be.."*** (Cl3A to C)

In clip C13A children have been given 2 orange rods to represent the 1 litre mark on a jug. They are asked to label other possible divisions through rod comparisons. First the white is named as one fortieth of two litres, red is then named as one twentieth of two litres, and pink as one tenth of 2 litres. One child sees the name one twentieth as meaning that it fits in 20 times, whereas another notices that the denominator has halved as the rod size has doubled. Naming the pink after the red allows her to extend her theory. The two justifications for the fraction name are presented simultaneously in the group. This is worth preserving in activities.

In clip Cl3B Three children discuss the name for orange and yellow placed end-to- end. One child has named it 1.5, but by reminding her that 10 oranges is 1, another shows that it must be 0.15 meaning one tenth and 5 hundredths. The children are able to confirm equivalent fraction and decimal names by referring back to previous comparisons.

Figure 2.6: Cl3A

Figure 2.7: CI3B

Figure 2.8: Cl3C

In clip Cl3C the children become aware that a fraction can have more than one name, I ask them to pull out names for one fraction and compare them. They quickly identify equivalent fractions where numerator and denominator are doubled and some suggest relationships of multiplying or dividing by other numbers. They continue a naming activity, but with this new awareness fresh in their minds. This group has decided to draw a vertical line between equivalent fractions to check if they fit this rule. Importantly, the names are still generated by comparison.

These activities do not always achieve what I was hoping for. Watching the clips shows that children are often only following one method of comparison at a time – even in groups they will agree another child's name and reason, but will return to their own route for establishing a name when it is their turn, so not really taking on another view point. A new task is needed requiring the children to establish the logical name by more than one comparison simultaneously to see how these operations are related to one another.

**And also, it must be, because (Cl3D to Gb)**

The activity generating equivalent names by empirical thought is now much shorter, with the children now being asked to find a 'route' from one name to another. They choose a name and explain why it can be substituted by another name. I believe this sequence illustrates an overlap between empirical phase and systematisation/mastery of structure.

In clip C13D Two children suggest names for rods using fractions, decimals and percentages simultaneously, reasoning with reference to previous rods. I think they are discovering new connections between names as they talk.

Figure 2.9: Cl3D

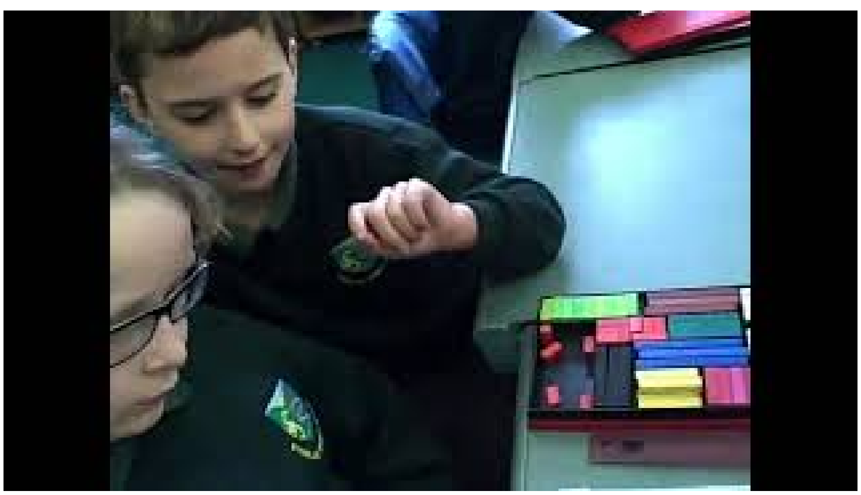


Figure 2.10: Cl3E

In clip Cl3E of the same activity, I notice how much the children's confidence with reasoning has grown. They have become used to explaining their thinking aloud.

Figure 2.11: Cl3F

I found clip Cl3F interesting because one child raises her own question about 4/4. Her partner explains it to her but she doesn't accept his explanation without finding her own 'route'. This strikes me as exactly the kind of discussion I want to encourage, where children can genuinely learn from each other.

The group in clip Cl3Ga have been generating equivalent names for fractions by comparison with other named rods, then checking that they fit their new theory that the numerator and denominator can be multiplied/divided by the same number to generate an equivalent fraction name. This seems to be close to mastery of structure, but to be sure I set out four fraction names and ask how they are related.

Figure 2.12: Cl3Gb

I think in clip Cl3Gb the child seems to have achieved some mastery of the situation. One girl is looking for a fraction they 'go back to'. It seems that she is able to see that "every element or group of elements is seen to potentially contain the infinite set of which it is part, as soon as the dynamic links between the elements have been noticed." (p.18 (Goutard, 2017) describing mastery.)

***Conclusions (Part 1)***

1. *The value of the empirical phase*

In these activities children use what they already know and apply it to a new operation. Goutard says children should be given what little they need and allowed to discover the rest. With relation to fractions, the little, seemed to be for example, the condensed ¾ to replace ¼ + ¼ + ¼. They were able to use adding, subtracting and multiplying fractions, to write mixed number and 'top-heavy' fractions and discover equivalent fraction names with no previous teaching of these areas. The value is that there is always an internal logic of the process to verify the name without needing a teacher or rule. Children need time to make their own discoveries, to see their own links and to build up a series of links which can later be drawn as evidence to reinforce understanding of structure. This stage is time-consuming, but necesary for deeper understanding. This explains to me why Goutard felt it was so important to start working on the four operations simultaneously from children's very first encounters with mathematics.

2. *How to move children towards systematisation and mastery of structure*
   By asking for names of fractions to be generated in a specific order or

> with some other limitation on naming, I aimed to put the children in a situation where structures became more apparent. Some children will just 'see' the structure, but the overlap with empiricism is important for them to remember how they arrived at their understanding so that they can re-create it at any time independently.

Explore Part 2 and watch the videos about the signs that an attempt at systematisation is being made and what triggers these attempts.

***Part 2 discussion***

*What are the signs that an attempt at systematisation is being made and what triggers these attempts?*

At this point in my research I re-read Goutard's chapter 'The Danger of Empiricism' and remembered that she was writing about the very youngest children. Above, I have illustrated the empirical phase with Years 2 and 3 and moving towards systematisation and mastery of structure with Years 4 and 5, but Goutard does not relate these phases to age and suggests that children can be at different stages with different operations at any one time. "The three phases that we have found in the process of structuration do not correspond to any chronological stages in the development of intelligence. It would be unhelpful to associate one age level with empirical investigation, another more advanced to the period of systematisation and so on…Most ten year old children have only fragmentary and empirical knowledge of mathematics because they have never been given a chance to explore it adequately. In contrast, children of 6 can, with a proper education, master structures. …There will be some overlapping of the three pedagogical phases …they may be at a more advanced stage with respect to additions and subtractions and be at the empirical stage only with respect to fractions, which form a more complicated set" p.28 (Goutard, 2017).

So I began to look for the three phases of working with Reception and Year 1 children, keen not to miss opportunities for moving beyond empiricism.

**Video clips Year 1, term 1. Writing addition statements: "I know it because" (Cl1A to C)**

Again, I needed an activity in which the children articulate their ideas to others allowing me to hear their reasoning. In this sequence of clips the Year 1 children are writing their own statements using the knowledge they have gained about the rods.

In clip Cl1A the child writes d = g + g on the interactive whiteboard then sets out rods to show her group what her writing means. The rods are used to explain her thinking. This seems to be empirical thought, but its value is in exploring use of the mathematical signs.

Again, clip Cl1B seems to demonstrate the same value of this empirical activity for learning to use signs. The child writes y = g + r. When she reads this to the group she says 'yellow plus green equals red'. When she asks the group if


Figure 2.13: Cl1A

Figure 2.14: Cl1B

they agree they say 'no'. She then tells them that she means 'yellow on its own and green and red together'. She places the rods in this arrangement. When she reads her statement again she reads the = as 'plus' but quickly corrects herself. This shows she is clear about what she wants to say, but it not yet sure of how the signs relate to it. I notice that she writes y = g +, stops to look up at the pictures of the rods, then nods before writing 'r'. She seems to be gathering empirical evidence.

Figure 2.15: CI1C

In clip Cl1C the child seems to have mastered the signs and be ready for more systematic work. He writes confidently, saying aloud what he means and what the symbols communicate. He asks the others if they agree showing that his written mathematics is a form of communication. He drags the rods into place in the same order he has written them, but I wonder if it is necessary for him to illustrate it with the rods; couldn't he use other reasoning to verify his statement and would this take him beyond the empirical phase? Are the rods keeping him too long at the empirical phase?

"..answers are not what matters most and children are capable of finding them mentally once teachers learn not to ascribe much importance to them but to watch the dynamics which serve as their basis instead" p.3 (Goutard, 2017)

This is what I am trying to find out from the child in the next clip – the dynamics which serves as the basis of his statement.

Although he uses the rods to explain how his signs should be interpreted, they do nothing to verify his statement about equivalent length because they are not accurately drawn or easy enough to position exactly. In terms of his understanding this doesn't seem to matter. I believe he knows enough about the relationship between blue and orange and between white and red to justify his writing. Notice that he smiles confidently and is not concerned that the rods don't fit – he just overlaps them slightly! This, along with his attempts at explaining his reasoning, seems to suggest that he is ready for more challenges which will take him beyond empirical thought.

***Conclusions***

The empirical phase is useful in confirming the child's intentions when using a mathematical sign, ensuring that we share an understanding of the situation it signifies. It is useful for the children to gather information which they can organise later, but there seems to be a point at which the child no longer needs the rods to prove their statements and at this point they need to be put in situations where they can explain their reasoning in other ways. The following clips show activities aimed at doing this. (Cl1 E to H)

Children are trying to find different ways of matching the length of the orange rod with other rods. I believe this work is at the empirical phase but already shows some evidence of systematisation. Substitution and the commutative nature of addition have been used in successive rows of rods showing that some level of awareness is present.

In clip Cl1F, the children are setting out partitions of the yellow rod and have been asked 'have you got them all?' One child begins a more systematic approach by grouping partitions by colour.

Again in response to the question 'do you think you have got them all?' groups set about organising rods into different groups and explain their flow of ideas.

Figure 2.16: CI1E

Figure 2.17: Cl1F

Figure 2.18: Cl1G:Pattern for the dark green (Year 1) [d]

Figure 2.19: Cl1H

Following Goutard's advice to move away from the rods in order to achieve mastery of structures, I try asking two children to tell me in clip Cl1H how many ways they could make yellow. Rather than remembering partitions, they seem to be reasoning about them using the commutative law and substitution as they go along: One suggests it could be done 'the other way round', another mentally substitutes red for two whites.

***Conclusions (Part2)***

In these examples it seems to be the questions 'have you got them all?", 'how can you be sure?' and 'is there a way of checking?' that prompt systematic work. Ideally, I would like these questions to come from the children, but this will require careful planning.

In an attempt to hear children articulate their understanding of an operation, I ask the children to write what they know about different rods (without the rods present), hoping that they will reveal any knowledge of structure they have gained. (Cl1I to Md)

Figure 2.20: Cl1I

In Cl1I the child uses her knowledge of subtraction as the inverse of addition to explain her statement.

In Cl1J she goes on to use substitution to explain her second statement – that orange can be replaced by p + d

The child in clip Cl1K uses substitution to cancel out letters in his equation, then is left with a commutative statement which he feels to too obvious to need explanation!

Using one of Goutard's activities, I then ask the children to re-write a statement, altering only one part, again hoping to see what structures they are aware of.

Figure 2.21: Cl1J

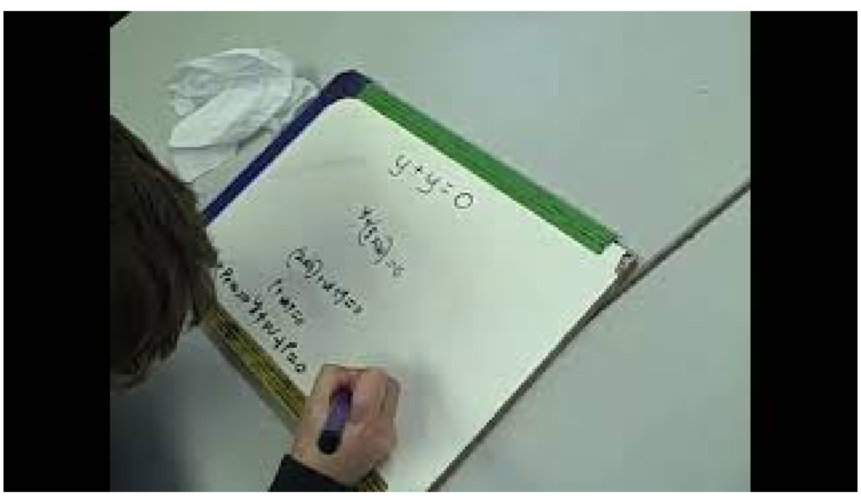
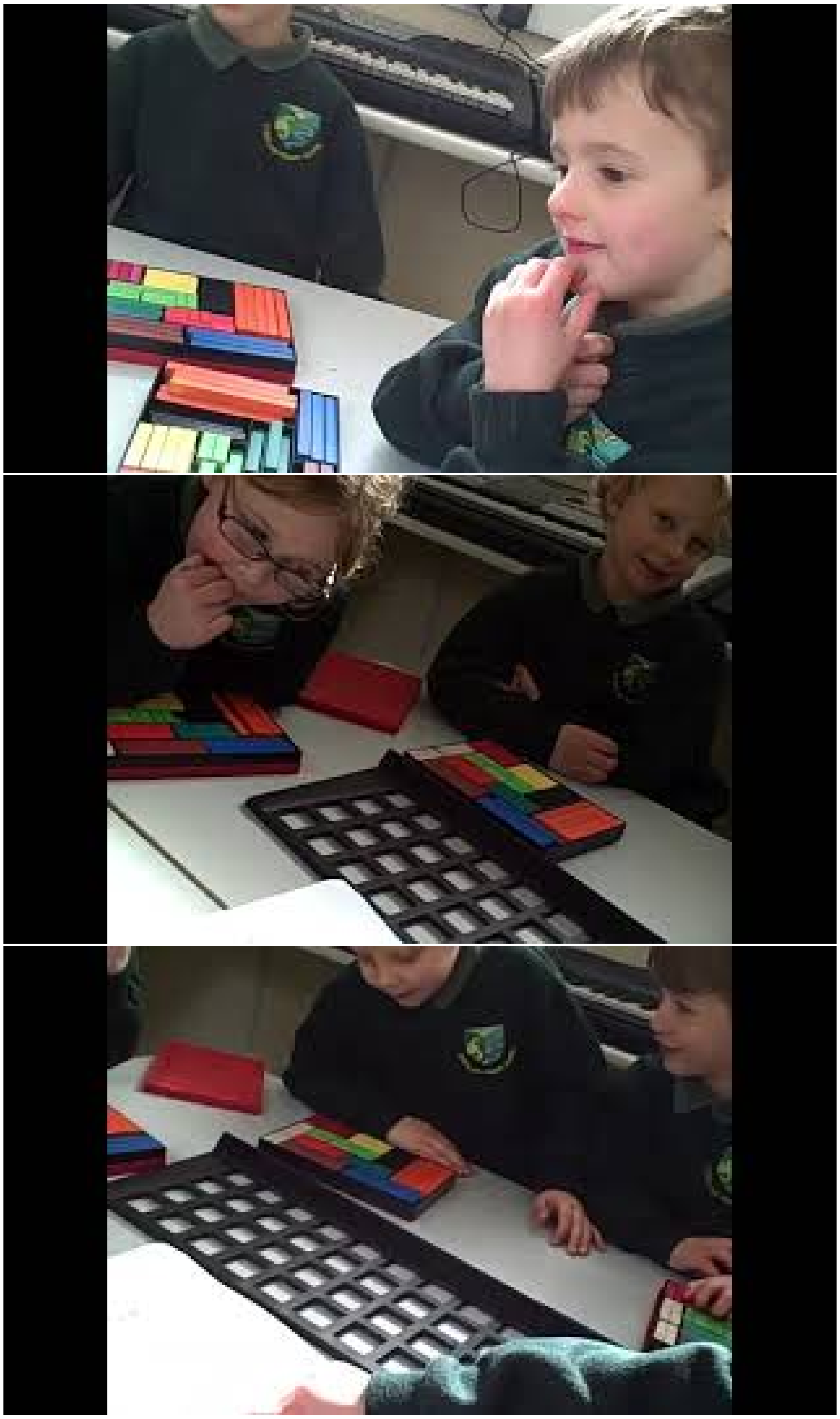



Figure 2.22: Cl1K

Figure 2.23: CI1L

In clip Cl1L children take turns to suggest equivalent expressions. They seem to show knowledge of addition as commutative and use substitution, but it would have been more useful to ask them to explain their statement aloud to hear their reasoning.

In this final sequence of clips I present the children with a series of equations, typical of those they had written for themselves in earlier lessons. Each equation is a transformation of the previous one. I was interested to see if the children's understanding of structure was sufficient to allow them to follow someone else's transformations. I try various approaches to encourage them to reason individually, with a partner and as a group. Ma, b, c, d



***Conclusion***

At this point believe I have seen that even the youngest children can move away from empirical thought if given the opportunity. This shift seems to take place when the rods are not present and children are in the position of transforming a statement, either orally or in writing, using what they know to be possible with an operation. This seems to explain the importance of independent writing by the children, not as a way of showing what they know, but as an opportunity to try out the structures they are beginning to see.

***Next steps***

At this point in my research, I have attempted to interpret Goutard's theory into classroom teaching and have seen what I believe to be examples of her three phases of working, even with the youngest children. I am beginning to understand her concerns that children are limited by too empirical a mode of thought, having seen evidence of systematisation and even mastery in some children after a very short time working with a new operation. Even after such in-depth study of the texts, I still have much more to learn, but I believe that it is only through classroom-based research that I can find out more about young children's remarkable abilities as mathematicians.

### 2.2.2 Interview with Pete Griffin

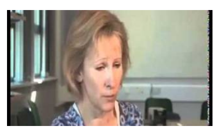


Figure 2.24: Pete Griffin and Caroline Ainsworth

### 2.2.3 Samples of children's writing

Goutard recommended allowing children time each day to write their own mathematical thoughts. Initially I was concerned that this would take up too much curriculum time and was unsure of the purpose of such writing. However, I now find this activity is central to children's learning for the following reasons:

- to discover relationships between operations
- to give opportunity for children to systematize and move towards mastery
- to learn the 'manufacturing secrets' of equations through transformation
- to gain flexibility, (leading to efficiency) in calculation strategies
- to allow children to pursue their own interests and investigations; to write maths for fun and the enjoyment of feeling in control of the subject

The following extracts of children's writing illustrate these points. They took place during free-writing time. You may like to print these out to aid discussion amongst colleagues.

#### Relationships between operations

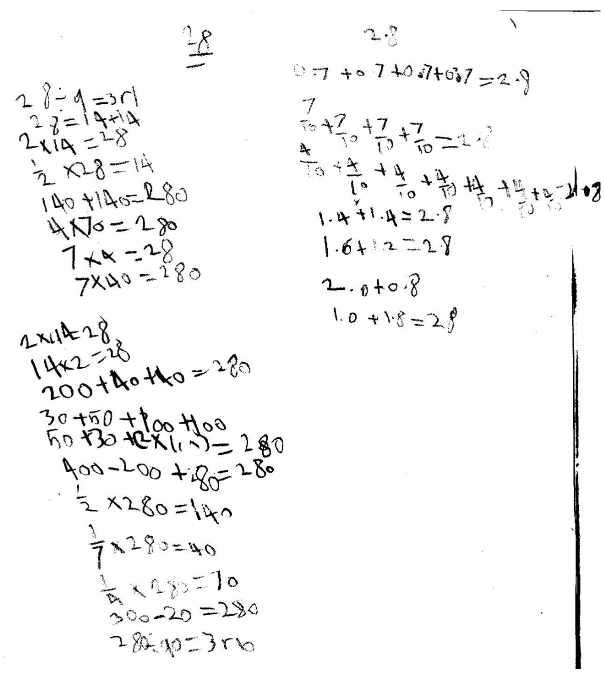


Figure 2.25: Year 2 - free writing after studying factors of 28

#### Systematization of facts

#### The 'manufacturing secrets' of equations - transformation

- Year 3 – written after playing a class game where the same equations is rewritten with only one term replaced each time. (3a)

#### Calculation strategies

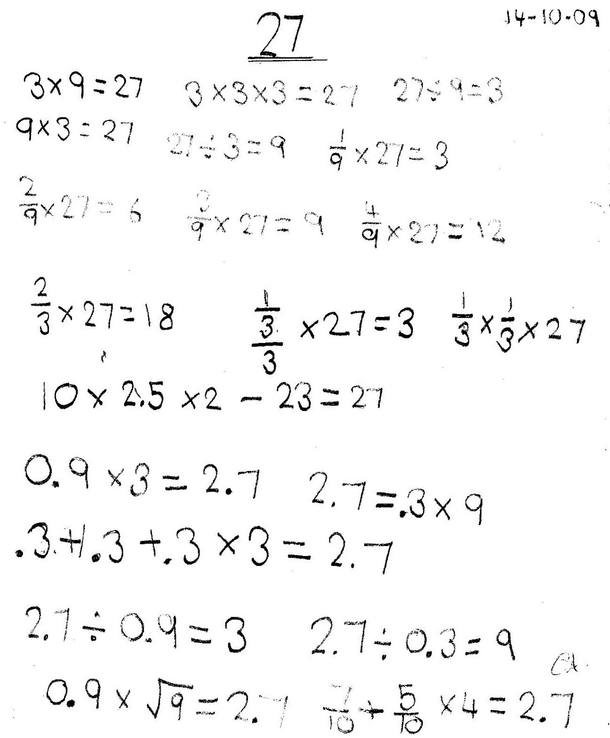
#### Pursuing interests and investigations - enjoying power over mathematics

## 2.3 Jenny Cane and Suzanne Spencer

Getting Started with Early Algebra

Experiences with Early Algebra

In these two articles, teachers describe how they are meeting the ambitious Key Stage 1 objectives of the new curriculum through mathematical writing. The 2013 curriculum requires learners 'to move fluently between representations of mathematical ideas.' (The National Curriculum in England, September 2013,


27 14-10-09

$3 \times 9 = 27$ $3 \times 3 \times 3 = 27$ $27 \div 9 = 3$

$9 \times 3 = 27$ $27 \div 3 = 9$ $\frac{1}{9} \times 27 = 3$

$\frac{2}{9} \times 27 = 6$ $\frac{3}{9} \times 27 = 9$ $\frac{4}{9} \times 27 = 12$

$\frac{2}{3} \times 27 = 18$ $\frac{\frac{1}{3}}{3} \times 27 = 3$ $\frac{1}{3} \times \frac{1}{3} \times 27$

$10 \times 2.5 \times 2 - 23 = 27$

$0.9 \times 3 = 2.7$ $2.7 = .3 \times 9$

$.3 + .3 + .3 \times 3 = 2.7$

$2.7 \div 0.9 = 3$ $2.7 \div 0.3 = 9$

$0.9 \times \sqrt{9} = 2.7$ $\frac{7}{10} + \frac{5}{10} \times 4 = 2.7$

Figure 2.26: Year 4 – written after study of 27

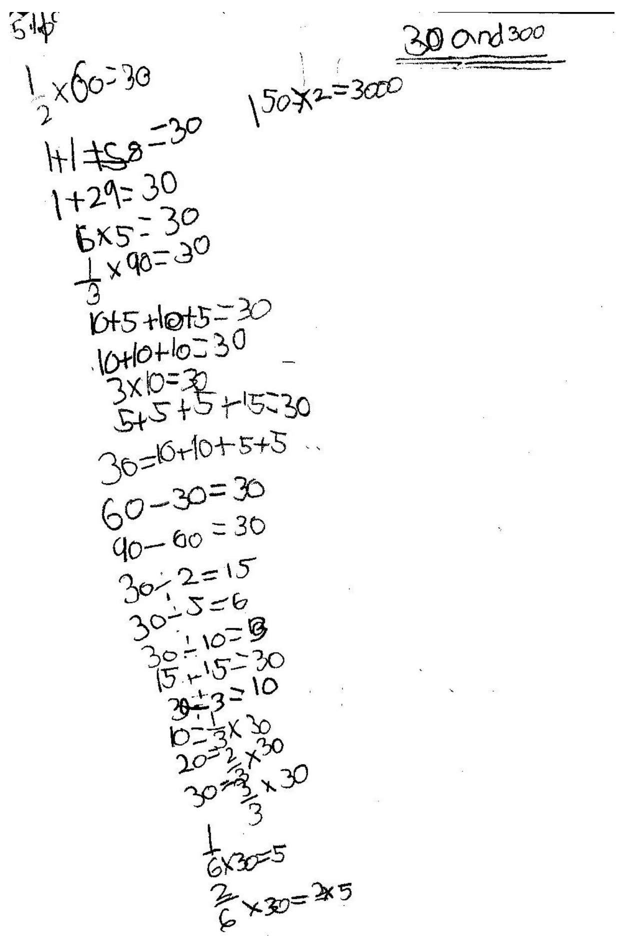



Figure 2.27: Year 3 – written after study of factors of 30

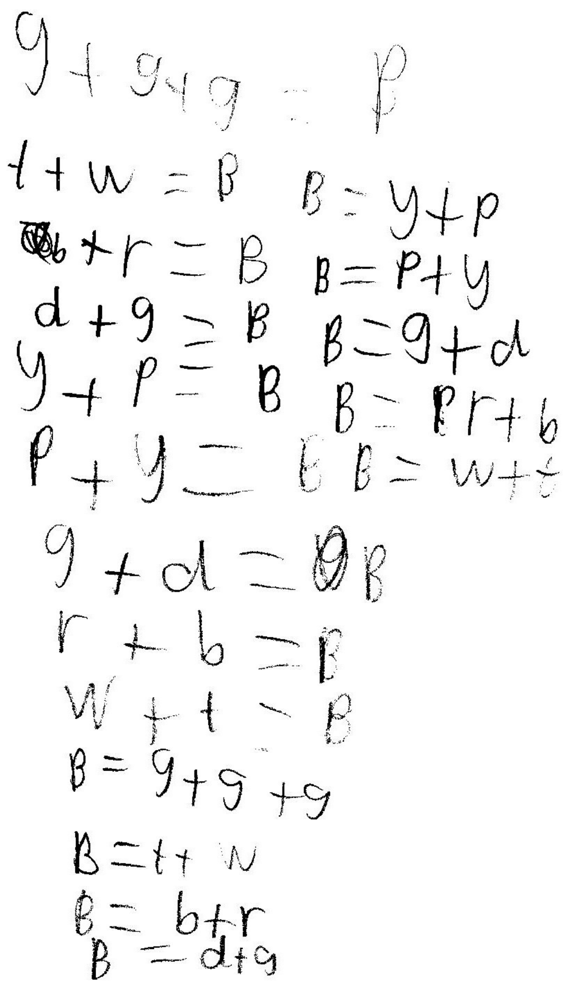


Figure 2.28: Year 1 – written after study of partitions of the blue rod

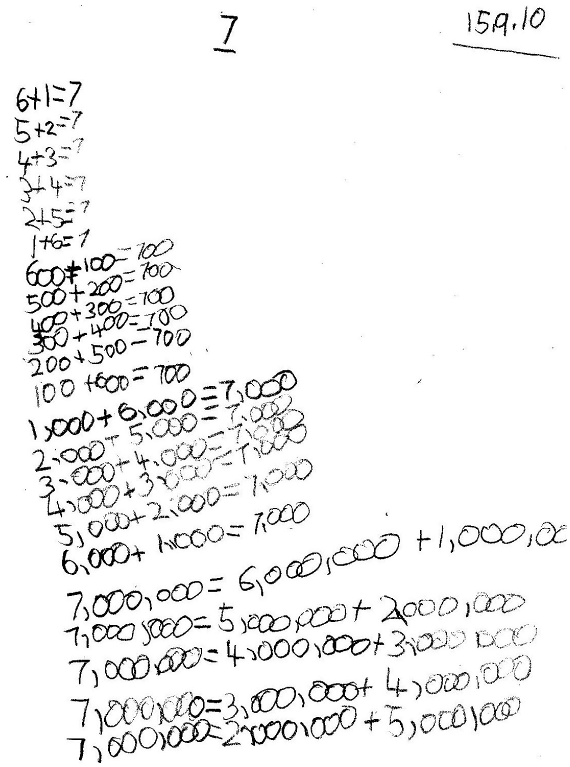



Figure 2.29: Year 2 – written after study of partitions of the black rod

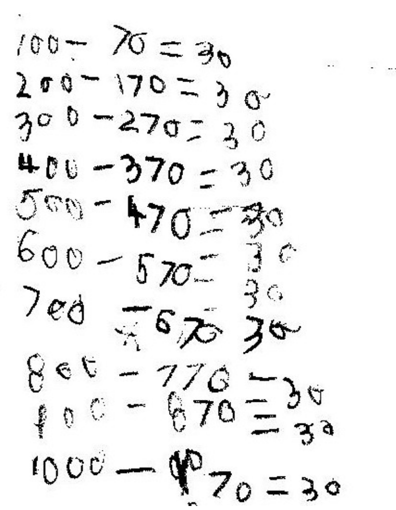



Figure 2.30: Year 2 – written after study of equivalent differences

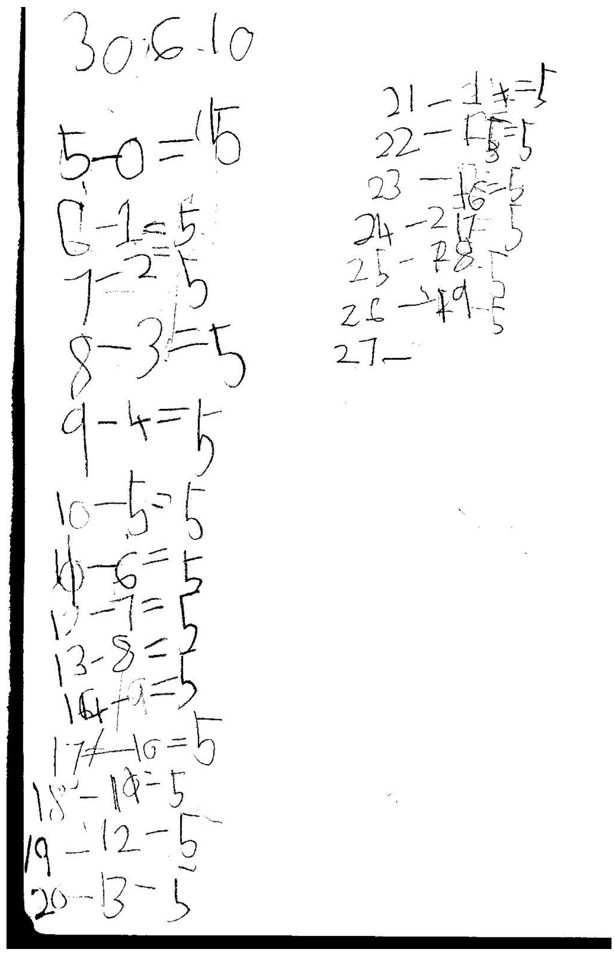


Figure 2.31: Year 1 – written after study of equivalent differences

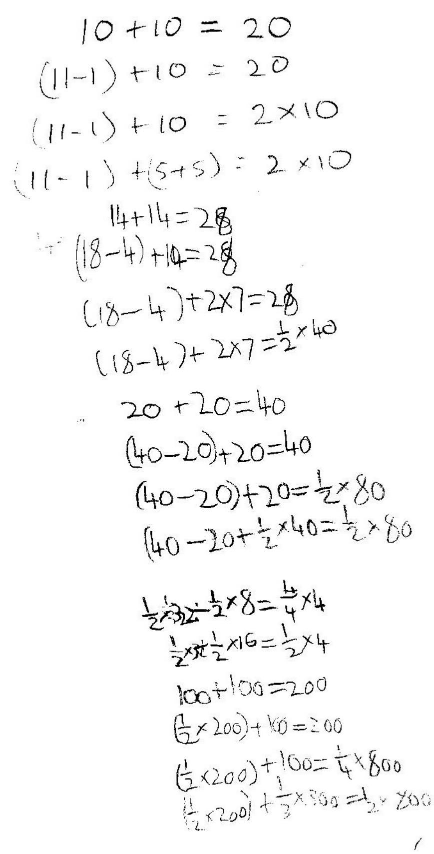


Figure 2.32: Year 3 – written after playing a class game where the same equations is rewritten with only one term replaced each time

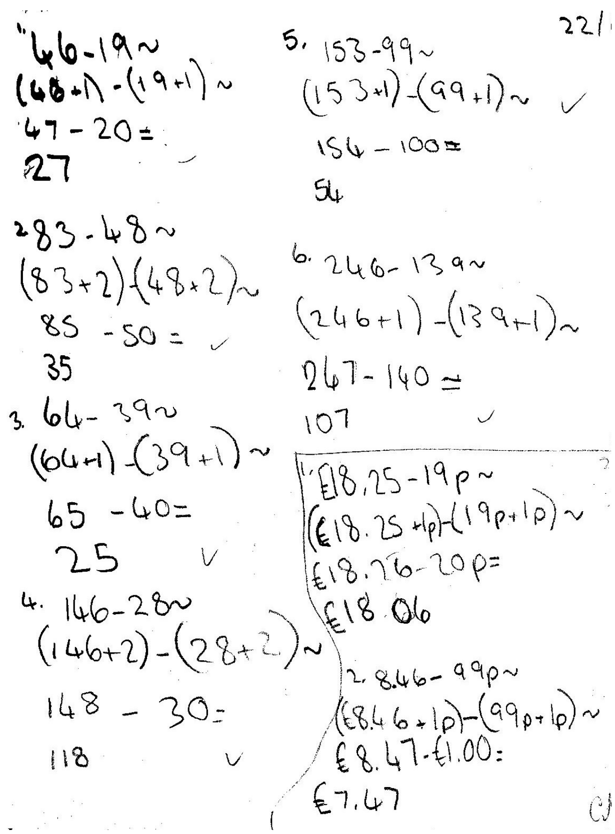



Figure 2.33: Year 5 – using awareness of equivalent differences in calculation

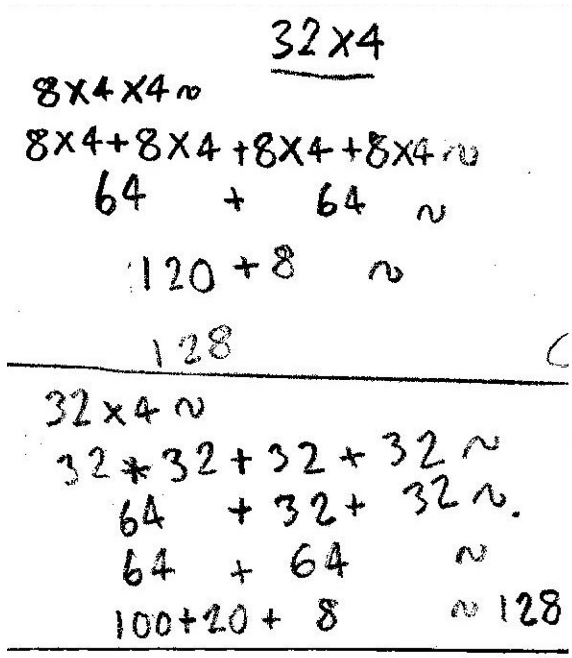



Figure 2.34: Year 5 – finding different ways to calculate multiplication

page 3.) Children now have to study all four arithmetic operations and fractions as operators for small numbers from Year 1

## 2.4 Selected Readings edited with Jim Thorpe

Working with the rods and why

The articles in this booklet bring together inspirational writings on the theory of reforming mathematics education together with articles by primary teachers who exemplify the Cuisenaire-Gattegno approach in practice.

Sixty years after Cuisenaire, Gattegno and Goutard embarked on this journey, new demands on mathematics teachers and new developments in conceptual mathematics and computer languages make reform both more urgent and more tractable.

The 2014 national curriculum is one of the first in the world to mandate that all four arithmetic operations and fractions as operators be studied from Year 1. Gattegno's textbooks propose an algebra of colour coded Cuisenaire rods to do just this.

In his Science of Education Gattegno proposed " subordinating teaching to learning" to harness mental powers present in every child. He set out a new role

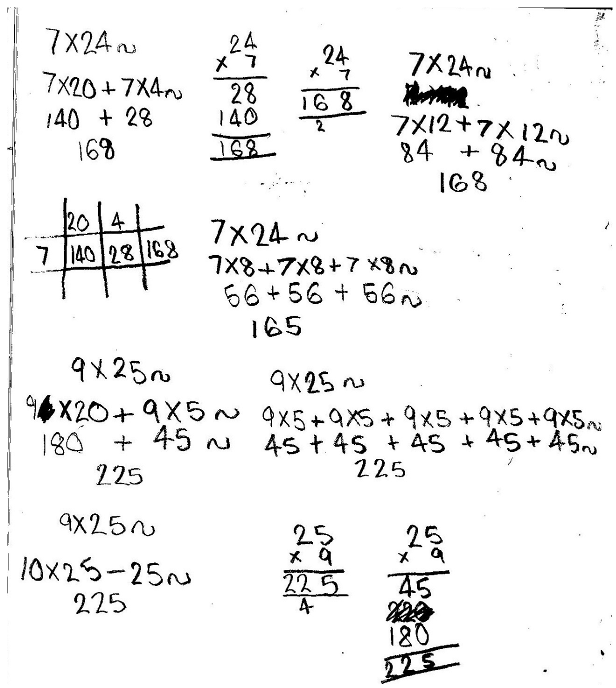



Figure 2.35: Year 5 – asked to express the same calculation in different ways

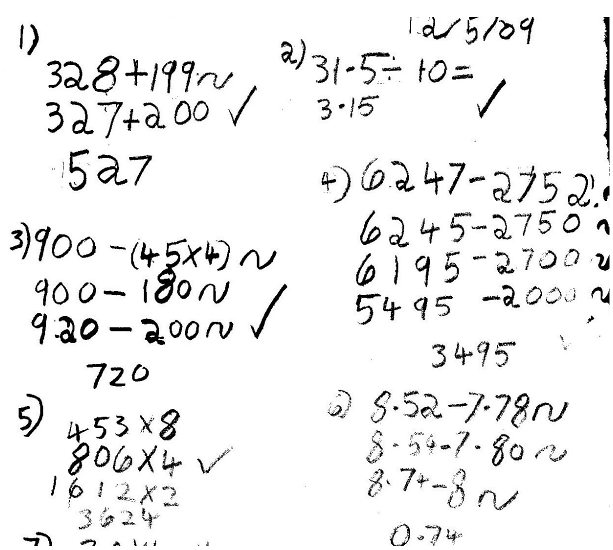



Figure 2.36: Year 6 – using equivalence to calculate

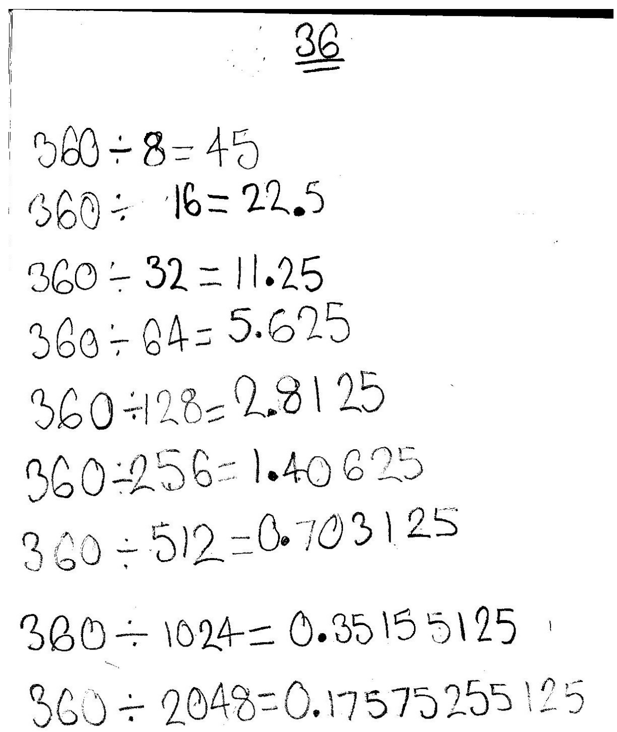



Figure 2.37: Year 4 – playing with halving

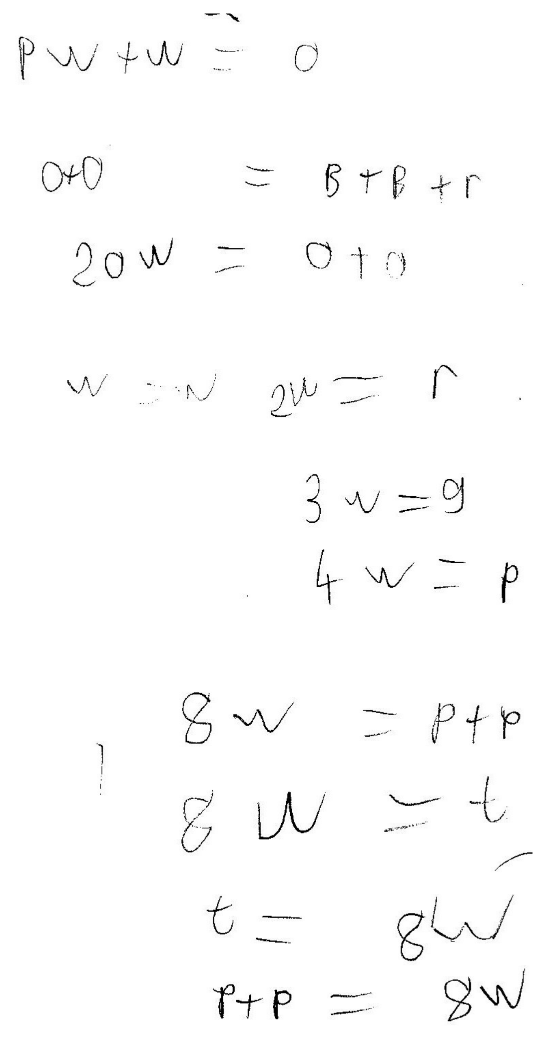



Figure 2.38: Year 1 – writing own ideas after free play with rods

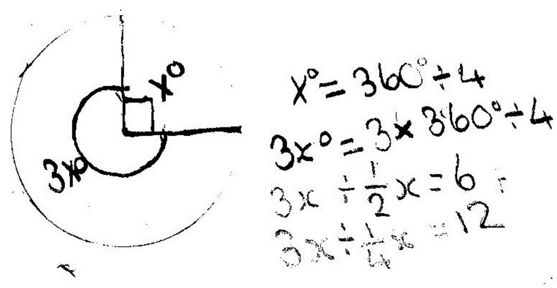



Figure 2.39: Year 5 - written after study of angles

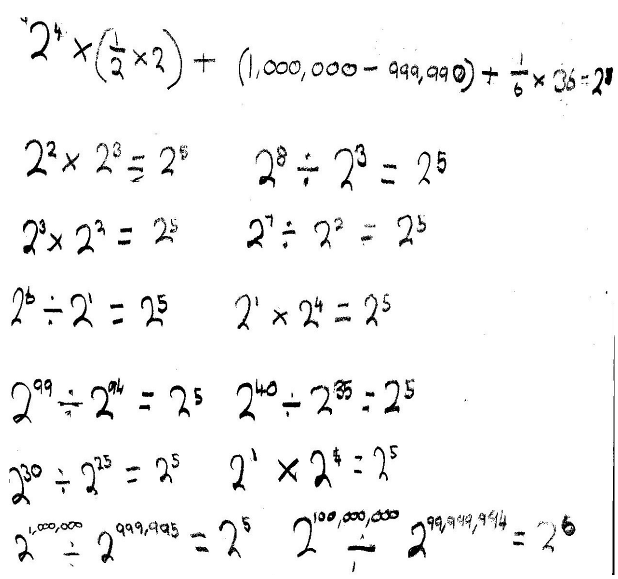



Figure 2.40: Year 5 – playing with powers (despite errors the child is working out a strategy for halving decimals)

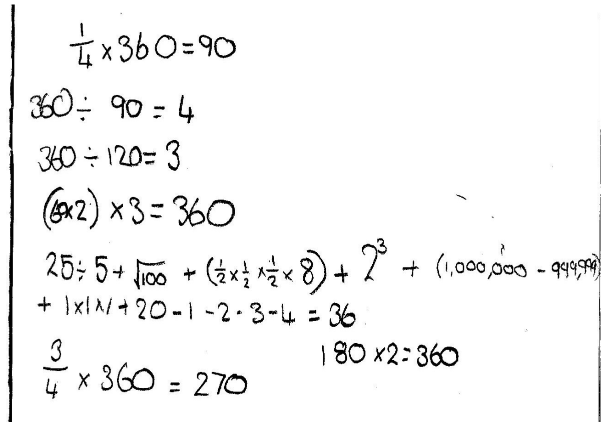



Figure 2.41: Year 5 – written after study of angles

for the teacher - to create game-like situations that the learner can experience mathematically, while supplying the labels and notations they cannot invent themselves. He proposed a way of talking about their learning in terms of awarenesses - which could be subconscious, conscious, named or categorized.

Madelaine Goutard illustrates this approach to lesson design in her article.

The articles from Jenny Cane and Caroline Ainsworth show how inspiring teaching can emerge from a close reading of Gattegno and Goutard's books. They contain open questions to guide learners exploring whole number and fractional relationships through permutations and combinations of Cuisenaire rods. Ian Benson discusses why these books differ from traditional textbooks and how this approach can be extended to further enrich school mathematics.

## 2.5 Meta-analysis with Bruce McCandliss and Nigel Marriott

Equational Reasoning: A Systematic Review of the Cuisenaire-Gattegno Approach

The Cuisenaire-Gattegno (Cui) approach to early mathematics uses colour coded rods of unit increment lengths embedded in a systematic curriculum designed to guide learners as young as age five from exploration of integers and ratio through to formal algebraic writing.

The effectiveness of this approach has been the subject of hundreds of investigations supporting positive results, yet with substantial variability in the nature of results across studies. Based on an historical analysis of one of the highest-fidelity studies Brownell, which estimated a treatment effect on equation reasoning with an effect size of 1.66, we propose that such variability may be related to different emphases on the use of the manipulatives or on the curriculum from which they came.

We conducted a systematic review and meta-analysis of Cui that sought to trace back to the earliest investigations of its efficacy. Results revealed the physical manipulatives component of the original approach (Cuisenaire Rods) have had greater adoption than efforts to retain or adopt curriculum elements from the Cuisenaire-Gattegno approach. To examine the impact of this, we extended the meta-analysis to index the degree to which each study of Cuisenaire Rods included efforts to align or incorporate curricular elements, practices, or goals with the original curriculum. Curriculum design fidelity captured a significant portion of the variability of efficacy results in the meta-analysis.

## 2.6 Longitudinal Study with Bruce McCandliss and Nigel Marriott

Interventions to improve equational reasoning: replication and extension of the Cuisenaire-Gattegno curriculum effect

In this paper we report on a controlled study with 120 students over the first two years of schooling contrasting the Cuisenaire-Gattegno curriculum approach vs. traditional rote learning on equational reasoning.

The ability to reason about equations in a robust and fluent way requires both instrumental knowledge of symbolic forms, syntax, and operations, as well as relational knowledge of how such formalisms map to meaningful relationships captured within mental models. Our systematic review of studies Section 2.5 contrasted the Cuisenaire-Gattegno (Cui) curriculum approach versus traditional rote schooling. It demonstrated the positive efficacy of pedagogies that focus on integrating these two forms of knowledge.

Here we seek to replicate and extend the most efficacious of these studies Brownell by implementing the curriculum to a high degree of fidelity, as well as capturing longitudinal changes within learners via a novel tablet-based assessment of accuracy and fluency with equational reasoning. We examined arithmetic fluency as a function of relational reasoning to equate initial performance across diverse groups and to track changes over four growth assessment points. Results showed that the intervention condition that stressed relational reasoning leads to advances in fluency for addition and subtraction with small numbers. We also showed that this intervention leads to changes in problem solving dispositions toward complex challenges, wherein students in the CUI intervention were more inclined to solve challenging problems relative to those in the control who gave up significantly earlier on multi-step problems.

This shift in disposition was associated with higher accuracy on complex equational reasoning problems. A treatment by aptitude interaction emerged for both arithmetic equation reasoning and complex multi-step equational reasoning problems, both of which showed that the intervention had greatest impact for children with lower initial mathematical aptitude. Two years of intervention contrast revealed a large effect (d =1) for improvements in equational reasoning for the experimental (CUI) group relative to control.

The strong replication and extension findings substantiate the importance of embedding these teaching aides within the theory grounded curricula that gave rise to them.

# Chapter 3

# Method of Approach

## 3.1 Co-design principles

We have pioneered a unique co-design approach to working with stakeholders - parents, pupils, teachers, academic researchers and government - on innovation in the school curriculum.

Figure 3.1 is a cartoon that illustrates our approach. It formed part of our annual report to stakeholders in 2008. The signpost points to the various Stanford and Cambridge University libraries where we conducted desk research. Teachers in Devon and Leicestershire and head teacher Steve Clarke are shown recording learners at work for subsequent analysis and INSET. The red box of Cuisenaire rods, together with our interactive Quicktime virtual manipulative ribbon, shows the objects we use to design "mathematising situations."

In the background are sketches of Dick Tizard (1917-2005) and Kristen Nygaard (1926-2002). This acknowledges their role in pioneering outreach, conceptual modelling and participatory design. Sarah Brown, the wife of the Prime Minister (2007-10), is photographed visiting our exhibition stand at the Labour Party conference. Gordon Brown's Policy Unit took a close interest in our work. Its head, Dan Corry, arranged for Sir Jim Rose, a member of the Williams Committee on Primary Mathematics, to visit Stockland School. Rose's positive assessment of the Cuisenaire-Gattegno approach is reproduced in the Front Matter.

Our approach to the participatory design of teaching and learning resources, including software, has its foundation in Gattegno's characterisation of the unfolding of mathematical activity. He illustrated this in Figure 3.2 (Catir and Gattegno, 1973) where he describes mathematical activity as unfolding through:

- *Action* using the number array, the set of fingers, rods ...

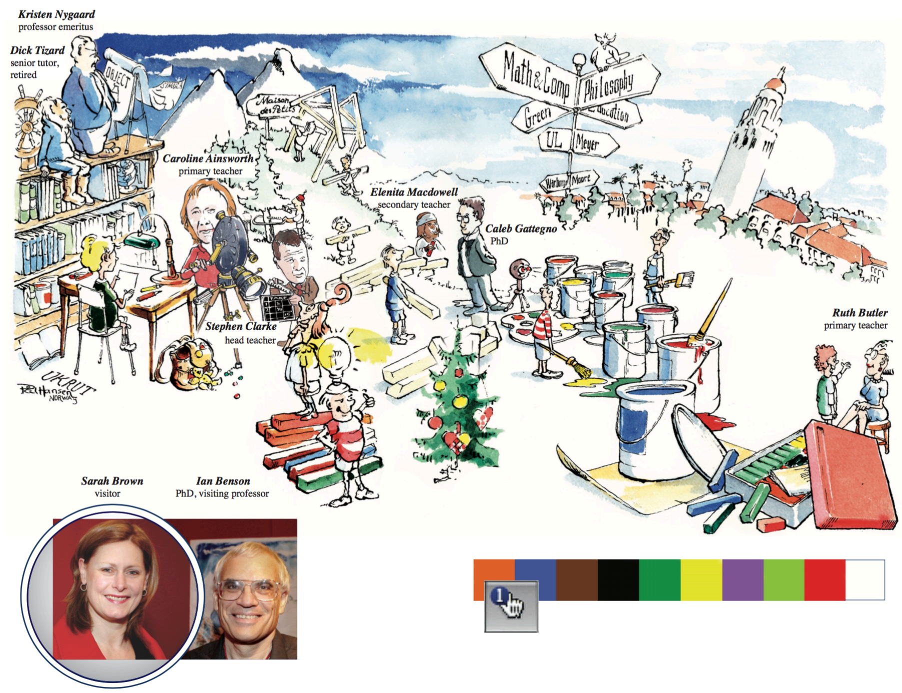



Figure 3.1: A virtual college

- *Virtual action* using imagery generated by the action
- *Speaking* language to describe the imagery
- *Writing* symbols and notation.

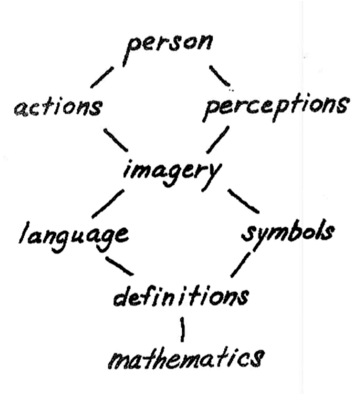



Figure 3.2: The unfolding of mathematical activity

### 3.1.1 Cycles of work

Our first step was to abstract Gattegno's process and embed it into a cycle of work (CoW). This meant that teachers could use his analysis to distinguish the stages in development of the learners mathematics from their own developing "meta-mathematical" awarenesses.

Figure 3.3 illustrates the dialogue between the teacher (on the left) and the learner (on the right) as a "conversation for action." This is a choreographed set of speech acts documented in (Winograd and Flores, 1986). We use the notation of the London Underground map Figure 3.3 to highlight decisions made by the teacher and learner from labels for activity that move the process on. Unlike Gattegno we permit the possibility that the learner is operating simultaneously at both imagining and recognising a mathematical situation.

The figures below are taken from (Benson, 2011) (pages 10 and 44). They were introduced at INSET sessions for the Leicestershire schools with the DfES Director of Strategy, Michael Stephenson and at a workshop with teachers at Stockland in June 2007.

## 3.2 Meta-mathematics

The next step was to develop a meta-mathematical account, in conceptual mathematics, that could succinctly model what the teachers were learning about how to think algebraically about the structure of number systems (Cheng, 2022)

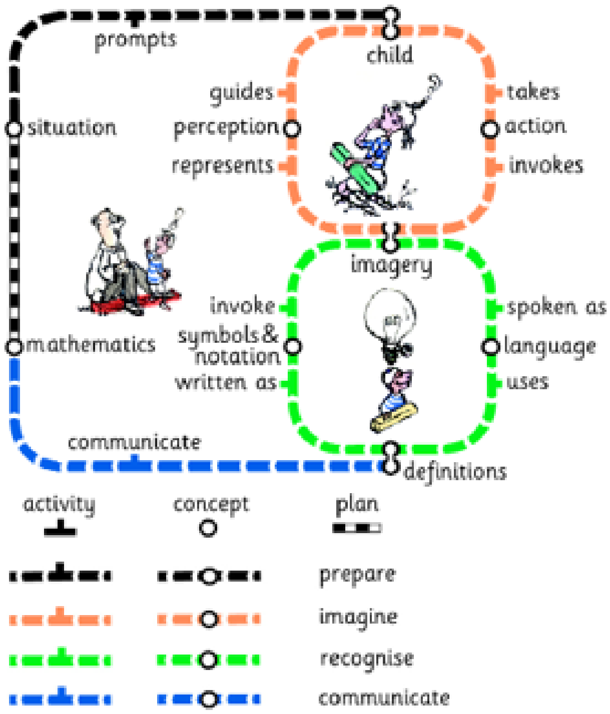



Figure 3.3: A cycle of learning and teaching

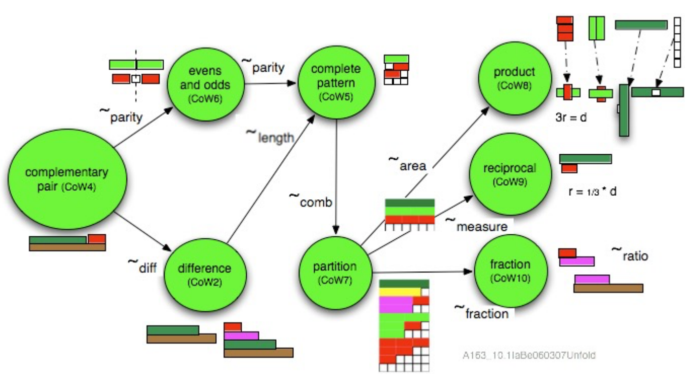



Figure 3.4: Cycles of work to cover primary maths

Ten cycles of work were abstracted from (Gattegno, 1986), 8 of which are recorded in the diagram, Figure 3.4. This shows the interdependencies between cycles as domains of reasoning. Each domain is a category of rod constructions with an organising equivalence relationship: length, parity etc (Cheng, 2022).

Early algebra is now seen as an important area of educational research and we were able to test our developing understanding with peer review at meetings of the ATM/MA Primary Expert group report by Fran Watson of nrich May 2016 and at several ATM conferences. Susan Empson et al are typical of this developing consensus when they write: *"We suggest that a model of the development of children's understanding of arithmetic that is based upon a concrete to abstract mapping is too simplistic. We propose instead that developing computational procedures based on relational thinking could effectively integrate children's learning of the whole-number and fraction arithmetic in elementary mathematics, in anticipation of the formalization of this thinking in algebra."* The Algebraic Nature of Fractions: Developing Relational Thinking in Elementary School , Susan B. Empson, Linda Levi, and Thomas P. Carpenter, in Early Algebraization, eds Jinfa Cai and Eric Knuth, Springer, 2011, p 411

Jenny Cane and Ian Benson elaborated one such approach to meta-mathematics at the Primary Expert Group in May 2016. We illustrated a unified pathway from primary mathematics to informatics (computer science), the natural sciences and secondary mathematics Figure 3.5.

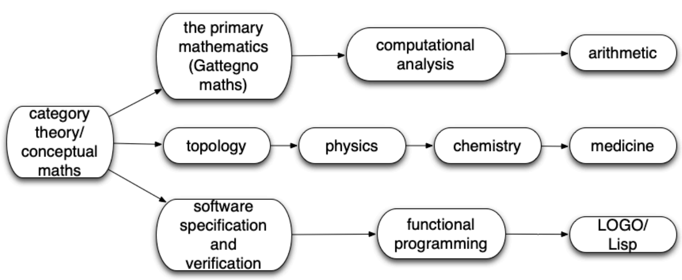



Figure 3.5: The Primary Mathematics (Curriculum Zero)

# Chapter 4

# Applications

Sociality's role is to study the teachers' professional development journeys, curate and co-produce software and workflows that address gaps and bottlenecks in understanding. Over nearly 20 years of work we have created many applications - ranging from virtual rods, puzzles and games (Whiteboard, Interactive Quicktime, HTML, iOS).

Three are illustrated here: an application by Nrich of the virtual rods, notHiding by Ian Benson and a tiling puzzle by Greg Gomberg. The puzzle was developed to stimulate discussion at the ATM conference Computational Thinking in Mathematics strand at the virtual conferences in 2021 and 2022. Here you can read a report and watch the proceedings.

## 4.1 Virtual rod environments

Once our own HTML and teacher developed whiteboard applets were superceded we created a webpage that linked to the nrich virtual rod environment and the public domain version of gattegno (formerly Numbers in Colour).

mathigon number bars are another virtual rod resource. Unlike nrich mathigon chose to label the rods with their number names when measured with a white rod.

## 4.2 Gattegno code pelmanism

*notHiding* is an iOS (iPhone and iPad) application for toddlers, pre-school children and their parents. It helps to improve your child's ability to take turns, learn, sort, recall, name and code Cuisenaire® colors, using capital or lower case letters.

The game can be played in one or two player mode to stimulate conversation and discussion about game strategy. The object of the game is to turnover cards to complete a pattern of matching pairs.

There is one level of difficulty on the iPhone and two on the iPad matching

- pairs of colour cards selected at random from a palette of 10 Cuisenaire® colors
- pairs of random capital letters
- pairs of minuscule letters
- pairs of matching capital and minuscule letters
- pairs of matching Gattegno colour codes and Cui colors

In single player mode the object is to uncover all the cards. This exercises memory and the ability to name the individual colours, letters and codes. Once this is mastered a more difficult challenge is to uncover all the cards with the smallest number of taps. The learner can keep count of how near they are to this target, and note how their score improves with practice.

In two player mode the object is to turn over more pairs than your partner. Players who successfully turn over a pair will take the next turn. The game keeps count of the score, and regulates turn taking.

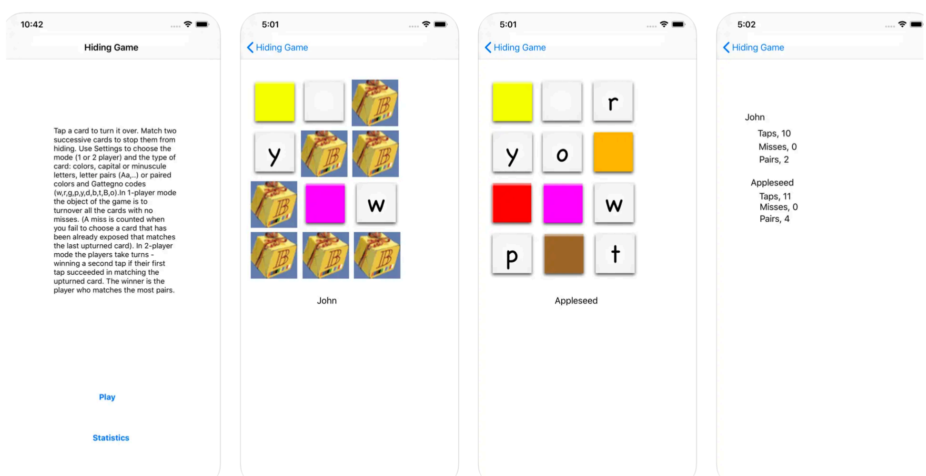



Figure 4.1: notHiding: an iOS app for parents and toddlers

## 4.3 Tiling Puzzle

Greg writes in this app “Computational Thinking has become a popular and apparently respectable concept, particularly in education, but I am not at all comfortable with it. It is a rebranding of a selection of mental tools (thought processes), and the danger is that enthusiasm for Computational Thinking will result in pupils being restricted in their thinking because they lack the tools that happen not to be in that selection. My personal view is that Computational Thinking is just a subset of mathematical thinking, but we do not need to argue about that. The point I want to make is that it does not cover all the tools one needs for computer coding, let alone for real life. Generations of social forces and poor education have made"Mathematics" a poisonous brand for pupils at school and for people who are highly educated in non-numerate disciplines. So I can see that inventing a new brand might seem a good idea. However, the descriptions I have found of Computational Thinking omit important parts of the mathematical thinking toolkit and that makes concentration on Computational Thinking dangerous.

“Here are some examples of an old family of puzzles you may have met before. I want to remind you of these - or introduce them if they are not familiar to you - because they provide an example where understanding requires the use of tools that are missing from the Computational Thinking kit - in this case, the concepts of symmetries and invariants - two sides of the same mathematical coin.

“If it were just a matter of puzzles then this would not be important. However, there are real-world practical problems whose abstract models are very like these puzzles, though more complicated. If, say, you accepted the task of coding an algorithm to help with one of those problems then you would probably start by making an abstract model similar to these puzzles and code your algorithm to solve that. Computational Thinking would certainly be necessary, but would not be sufficient. If you didn't use tools that are missing from Computational Thinking then the success of your algorithm would be a matter of luck.

“A well-rounded syllabus ought also to include other, less mathematical modes of thought. We need people to grow up with a better understanding of Scientific Thinking, Logical Thinking, Statistical and Probabilistic Thinking, Ethical Thinking, Thinking about Risks, Skeptical Thinking, Historical Thinking and so on. Overselling"Computational Thinking" certainly threatens the rest of mathematical thinking, but it looks likely to threaten these other capabilities as well."

# Chapter 5

# Conclusion

This book is a work in progress. If you have any thoughts and comments please email ian dot benson at cs dot stanford dot edu.

We look forward to hearing from parents, teachers and learners.

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